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# ELECTRO–ELASTIC INTERACTION BETWEEN A SCREW DISLOCATION AND AN ELLIPTICAL INHOMOGENEITY IN PIEZOELECTRIC MATERIALS

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**Abstract**—This article is concerned with the electro–elastic interaction between a dislocation and an elliptical piezoelectric inhomogeneity in an infinite piezoelectric matrix. The matrix is subjected to remote antiplane shear and inplane electric fields. The *explicit* expressions of the complex potentials are derived in both the inhomogeneity and the surrounding matrix using conformal mapping and the perturbation techniques. The results reveal that when the inhomogeneity reduces to a cavity, the electric field strength in both the cavity and the matrix is not affected by the dislocation. In addition, the results also show that the electric field strength is uniform in the cavity. In the case of a slit crack, the electric field strength in the matrix becomes uniform along the slit and in the matrix, while the stress and the electric displacements show the traditional square root singularity at the crack tip. © 1998 Elsevier Science Ltd.

## 1. INTRODUCTION

Due to their intrinsic electro–mechanical coupling behaviour, piezoelectric materials are widely used as sensors and actuators in the technologies of smart materials. Significant progress has recently been made in the development of these materials as a result of the extensive research efforts of the scientific and industrial communities.

It is well known that defects, such as dislocations, cracks, cavities and inclusions, can greatly influence the performance of piezoelectric devices. A thorough understanding of the coupled electro–mechanical behaviour of these devices requires accurate knowledge of both the electric and the mechanical fields produced by these defects.

A great deal of work had been conducted on the interaction between the electric and elastic fields induced by different defects in a piezoelectric material [see Parton (1976); Deeg (1980); McMeeking (1987); Sosa and Pak (1990); Kuo and Barnett (1991); Pak (1990a, b; 1992); Wang (1992); Chen (1993); among others]. Some important solutions have been derived e.g. Deeg (1980) examined the effect of a dislocation, a crack and an inclusion upon the coupled response of piezoelectric solids, while Pak (1990a) obtained closed-form solutions for a screw dislocation in a piezoelectric solid. Zhang and Tong (1996) formulated the mechanical and electric fields around an elliptic cylindrical cavity in a piezoelectric material under remote antiplane shear and inplane electric fields. With the extended eight-dimensional formalism developed by Lothe and Barnett (1976), Kuo and Barnett (1991) and Suo *et al.* (1992) studied the singularities of interfacial cracks in bonded anisotropic piezoelectric media. More recently, Chung and Ting (1996) considered an elliptical inclusion embedded in a piezoelectric matrix. An extensive review concerning the advances in piezoelectric solids with defects can be found in a recent paper by Sosa and Khutoryansky (1996).

In this paper, we complement the earlier works by examining the electro–elastic interaction effects between a screw dislocation and an elliptical piezoelectric inhomogeneity in an infinite piezoelectric matrix. The matrix is subjected to remote antiplane shear and inplane electric field. The analysis is based upon the use of conformal mapping and the perturbation method. Following this brief introduction, we state the problem and outline

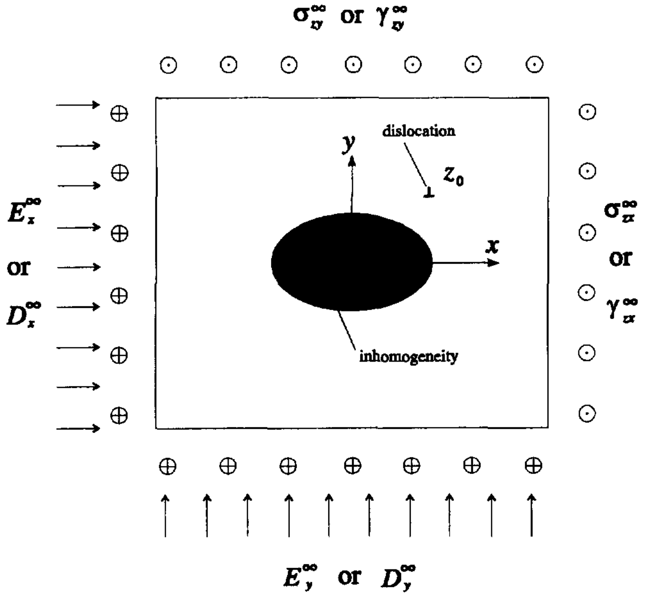


Fig. 1. A schematic of the electro-elastic interaction between a screw dislocation and an elliptical inhomogeneity in a piezoelectric material.

the basic field equations and continuity conditions. In Section 3, the general series solutions for the potentials, both inside and outside the inhomogeneity, are derived *explicitly*. In Section 4, closed-form solutions are obtained for a screw dislocation with a circular piezoelectric inhomogeneity. In Section 5, two special interaction problems are examined: (i) the elastic interaction between a dislocation and an elliptical inclusion, and (ii) the electro-elastic interaction between a dislocation and an elliptical cavity. The field solutions are given in closed forms. For the former problem, our solution is shown to be identical to that obtained previously by Sendeckyj (1970). For the latter one, it is found that the electric field strength, both inside and outside the cavity, is not influenced by the dislocation and is uniform when the cavity reduces to a slit crack. Finally, the paper is concluded in Section 6.

## 2. BASIC EQUATIONS

In a linear piezoelectric medium, the governing field equations and constitutive relations at constant temperature can be expressed as

$$\sigma_{ij,i} = 0, \quad D_{i,i} = 0 \tag{1}$$

$$\sigma_{ij} = c_{ijkl}u_{k,l} - e_{kij}E_k, \quad D_i = e_{ikl}u_{k,l} - \varepsilon_{ik}E_k \tag{2}$$

where $\sigma_{ij}$, $u_i$, $D_i$ and $E_i$ are stress, displacement, electric displacement and electric fields, respectively. $c_{ijkl}$, $e_{kij}$ and $\varepsilon_{ij}$ are the corresponding elastic, piezoelectric and dielectric constants, which satisfy the following symmetry relations:

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}, \quad e_{kij} = e_{kji}, \quad \varepsilon_{ij} = \varepsilon_{ji}.$$

Let us consider an unbounded piezoelectric medium which contains an isolated singularity and an elliptical piezoelectric inhomogeneity, subject to the uniform remote mechanical and electric loads shown in Fig. 1. Both the inhomogeneity and the matrix are assumed to be transversely isotropic, while the singularity and the inhomogeneity are infinitely extended in a direction perpendicular to $xy$-plane. The inhomogeneity is assumed to be perfectly bonded with the matrix and there are no concentrated forces and free charges lying at the interface. The singularity may be a line dislocation, a line force or a line charge.

In our study, the singularity will be considered as a screw dislocation located at point $(x_0, y_0)$ in the matrix with the Burgers vector given as $b_z$. The regions occupied by the matrix and the inhomogeneity are referred to as $\Omega_1$ and $\Omega_2$, respectively.

For the present problem, the anti-plane displacement $w$ is coupled with the in-plane electric field $E_x$ and $E_y$. They are independent of the longitudinal coordinate $z$, such that $w = w(x, y)$, $E_x = E_x(x, y)$ and $E_y = E_y(x, y)$. Then, the governing field equations (1) and the constitutive relations (2) reduce to

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0, \quad \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = 0 \tag{3}$$

$$\left.\begin{aligned} \sigma_{zx} &= c_{44}\frac{\partial w}{\partial x} - e_{15}E_x, \quad \sigma_{zy} = c_{44}\frac{\partial w}{\partial y} - e_{15}E_y \\ D_x &= e_{15}\frac{\partial w}{\partial x} + \varepsilon_{11}E_x, \quad D_y = e_{15}\frac{\partial w}{\partial y} + \varepsilon_{11}E_y \end{aligned}\right\}. \tag{4}$$

Substituting eqn (4) into eqn (3) and noting that $E_i = -\phi_{,i}$ where $\phi(x, y)$ is the electric potential, we have

$$\begin{aligned} c_{44}\nabla^2 w + e_{15}\nabla^2\phi &= 0 \\ e_{15}\nabla^2 w - \varepsilon_{11}\nabla^2\phi &= 0 \end{aligned} \tag{5}$$

where $\nabla^2$ is the two-dimensional Laplacian operator. It is easy to show that eqn (5) can be satisfied automatically, if $w$ and $\phi$ are chosen as the real parts of the analytical functions $\Psi(z)$ and $\Phi(z)$, such that:

$$\begin{aligned} w &= \frac{1}{2c_{44}}[\Psi(z) + \overline{\Psi(z)}] \\ \phi &= \frac{1}{2\varepsilon_{11}}[\Phi(z) + \overline{\Phi(z)}] \end{aligned} \tag{6}$$

where $z = x + iy$ is the complex variable and the overbar refers to the complex conjugate. Hence, the electric field strength, the electric displacements and the stresses can be expressed as

$$\begin{aligned} E_x - iE_y &= -\frac{1}{\varepsilon_{11}}\Phi'(z), \quad D_x - iD_y = \frac{e_{15}}{c_{44}}\Psi'(z) - \Phi'(z), \\ \sigma_{zx} - i\sigma_{zy} &= \Psi'(z) + \frac{e_{15}}{\varepsilon_{11}}\Phi'(z) \end{aligned} \tag{7}$$

where prime denotes the derivatives with respect to the arguments. Using eqn (7), the resultant force $T$ and the resultant normal component $S$ of the electric displacement along any arc $AB$ can be calculated as

$$\begin{aligned} T &= \int_A^B (\sigma_{zx}\,\mathrm{d}y - \sigma_{zy}\,\mathrm{d}x) = \frac{i}{2}\left\{[\overline{\Psi(z)} - \Psi(z)]_A^B + \frac{e_{15}}{\varepsilon_{11}}[\overline{\Phi(z)} - \Phi(z)]_A^B\right\} \\ S &= \int_A^B (D_x\,\mathrm{d}y - D_y\,\mathrm{d}x) = \frac{i}{2}\left\{\frac{e_{15}}{c_{44}}[\overline{\Psi(z)} - \Psi(z)]_A^B - [\overline{\Phi(z)} - \Phi(z)]_A^B\right\} \end{aligned} \tag{8}$$

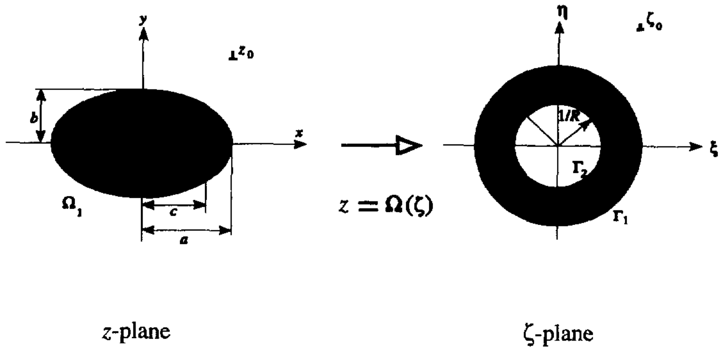


Fig. 2. A schematic of conformal mapping used.

where $[\,]_A^B$ represents the change in the bracketed function going from point $A$ to point $B$ along the arc.

Let us now introduce the following mapping function

$$z = \Omega(\zeta) = \frac{c}{2}[R\zeta + (R\zeta)^{-1}], \quad R\zeta = \frac{1}{c}[z + (z^2 - c^2)^{1/2}] \tag{9}$$

with

$$\zeta = \xi + i\eta, \quad c = (a^2 - b^2)^{1/2} = a(1-\varepsilon^2)^{1/2}$$
$$R = \left(\frac{a+b}{a-b}\right)^{1/2} = \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{1/2}, \quad \varepsilon = \frac{b}{a} \tag{10}$$

where $2a$ and $2b$ are the major and minor diameters of the elliptical inhomogeneity. This mapping function transforms region $\Omega_1$ of the $z$-plane into the exterior region of the unit circle $\Gamma_1$ ($\rho = 1$) in the transformed $\zeta$-plane. It also transforms region $\Omega_2$ into the annular region between the unit circle $\Gamma_1$ and a circle $\Gamma_2$ of radius $\rho = 1/R$ representing a cut from $-c$ to $+c$ in the $z$-plane, see Fig. 2. With the mapping function (9), eqns (6) and (8) can be rewritten in the $\zeta$-plane as

$$w = \frac{1}{2c_{44}}[\Psi(\zeta) + \overline{\Psi(\zeta)}]$$
$$\phi = \frac{1}{2\varepsilon_{11}}[\Phi(\zeta) + \overline{\Phi(\zeta)}] \tag{11}$$

and

$$T = \frac{i}{2}\left\{[\overline{\Psi(\zeta)} - \Psi(\zeta)]_A^B + \frac{e_{15}}{\varepsilon_{11}}[\overline{\Phi(\zeta)} - \Phi(\zeta)]_A^B\right\}$$
$$S = \frac{i}{2}\left\{\frac{e_{15}}{c_{44}}[\overline{\Psi(\zeta)} - \Psi(\zeta)]_A^B - [\overline{\Phi(\zeta)} - \Phi(\zeta)]_A^B\right\} \tag{12}$$

where $\Psi(\zeta)$ and $\Phi(\zeta)$ imply $\Psi[\Omega(\zeta)]$ and $\Phi[\Omega(\zeta)]$, respectively. By applying the perturbation techniques adopted by Stagni (1982) for isotropic elasticity and by Hwu and Yen (1993) for anisotropic elasticity, the general field potentials (11) for the inhomogeneity problem can now be written as

$$\left.\begin{aligned} w_1 &= \frac{1}{2c_{44}^1}[\Psi_0(\zeta)+\overline{\Psi_0(\zeta)}+\Psi_1(\zeta)+\overline{\Psi_1(\zeta)}] \\ \phi_1 &= \frac{1}{2\varepsilon_{11}^1}[\Phi_0(\zeta)+\overline{\Phi_0(\zeta)}+\Phi_1(\zeta)+\overline{\Phi_1(\zeta)}] \end{aligned}\right\} \quad \zeta\in\Omega_1 \tag{13}$$

and

$$\left.\begin{aligned} w_2 &= \frac{1}{2c_{44}^2}[\Psi_2(\zeta)+\overline{\Psi_2(\zeta)}] \\ \phi_2 &= \frac{1}{2\varepsilon_{11}^2}[\Phi_2(\zeta)+\overline{\Phi_2(\zeta)}] \end{aligned}\right\} \quad \zeta\in\Omega_2 \tag{14}$$

where the subscripts (or superscripts) 1 and 2 denote the matrix $\Omega_1$ and the inhomogeneity $\Omega_2$, respectively. The functions $\Psi_0$ and $\Phi_0$, which represent the field potentials associated with the unperturbed mechanical and electric fields, are holomorphic in the entire domain except at the singular points. The functions $\Psi_1$ and $\Phi_1$ (or $\Psi_2$ and $\Phi_2$) are the field potentials related to the perturbed field in the matrix (or inhomogeneity) and are holomorphic in region $\Omega_1$ (or $\Omega_2$).

The assumption of perfect bonding and that of no free charges and forces along the interface between regions $\Omega_1$ and $\Omega_2$ imply the continuity of displacement, electric potential, traction and normal components of the electric displacement across the elliptical interface. These conditions can be expressed as

$$w_1 = w_2, \quad \phi_1 = \phi_2, \quad T_1 = T_2, \quad S_1 = S_2 \quad \text{on } \Gamma_1 \quad (\zeta = \sigma = e^{i\theta}). \tag{15}$$

Substituting eqns (13) and (14) into eqn (15) yields

$$\mu_1[\Psi_0(\sigma)+\overline{\Psi_0(\sigma)}+\Psi_1(\sigma)+\overline{\Psi_1(\sigma)}] = \Psi_2(\sigma)+\overline{\Psi_2(\sigma)} \tag{16a}$$

$$\mu_2[\Phi_0(\sigma)+\overline{\Phi_0(\sigma)}+\Phi_1(\sigma)+\overline{\Phi_1(\sigma)}] = \Phi_2(\sigma)+\overline{\Phi_2(\sigma)} \tag{16b}$$

$$\begin{aligned}&[\overline{\Psi_0(\sigma)}-\Psi_0(\sigma)+\overline{\Psi_1(\sigma)}-\Psi_1(\sigma)]+\alpha_1[\overline{\Phi_0(\sigma)}-\Phi_0(\sigma)+\overline{\Phi_1(\sigma)}-\Phi_1(\sigma)] \\ &\qquad = [\overline{\Psi_2(\sigma)}-\Psi_2(\sigma)]+\alpha_2[\overline{\Phi_2(\sigma)}-\Phi_2(\sigma)]\end{aligned} \tag{16c}$$

$$\begin{aligned}&\beta_1[\overline{\Psi_0(\sigma)}-\Psi_0(\sigma)+\overline{\Psi_1(\sigma)}-\Psi_1(\sigma)]-[\overline{\Phi_0(\sigma)}-\Phi_0(\sigma)+\overline{\Phi_1(\sigma)}-\Phi_1(\sigma)] \\ &\qquad = \beta_2[\overline{\Psi_2(\sigma)}-\Psi_2(\sigma)]-[\overline{\Phi_2(\sigma)}-\Phi_2(\sigma)]\end{aligned} \tag{16d}$$

where

$$\begin{aligned}&\mu_1 = c_{44}^2/c_{44}^1, \quad \mu_2 = \varepsilon_{11}^2/\varepsilon_{11}^1, \quad \alpha_1 = e_{15}^1/\varepsilon_{11}^1, \quad \alpha_2 = e_{15}^2/\varepsilon_{11}^2 \\ &\beta_1 = e_{15}^1/c_{44}^1, \quad \beta_2 = e_{15}^2/c_{44}^2.\end{aligned} \tag{17}$$

In addition, the following conditions must be satisfied on $\Gamma_2$

$$\Psi_2(\sigma/R) = \Psi_2(\bar{\sigma}/R), \quad \Phi_2(\sigma/R) = \Phi_2(\bar{\sigma}/R) \tag{18}$$

since the points $\sigma/R$ and $\bar{\sigma}/R$ correspond to the same points of the cut from $-c$ to $+c$ in the $z$-plane. Noting that the remote mechanical and electric loads are uniform, the unperturbed solution potentials $\Psi_0$ and $\Phi_0$ with a screw dislocation located at point $z_0 = \Omega(\zeta_0)$ can be easily given as

$$\Psi_0(z) = \Psi_0(\zeta) = \frac{c_{44}^1 b_z}{2\pi i}\ln(z-z_0)+p_0 z$$
$$= \frac{c_{44}^1 b_z}{2\pi i}\ln[\Omega(\zeta)-\Omega(\zeta_0)]+p_0\Omega(\zeta)$$
$$\Phi_0(z) = \Phi_0(\zeta) = q_0 z = q_0\Omega(\zeta) \tag{19}$$

where the Burgers vector $b_z$ is a real number, $p_0$ and $q_0$ are complex constants which can be determined from the mechanical and electric loading conditions at infinity and can thus be taken as the remote equivalent mechanical and electric fields, respectively. There are four possible combinations of remote mechanical and electric loadings (Pak, 1990b):

Case 1: remote mechanical strains $\gamma_{zx}^{\infty}$, $\gamma_{zy}^{\infty}$ and remote electric field strength $E_x^{\infty}$ and $E_y^{\infty}$;

Case 2: remote mechanical stresses $\sigma_{zx}^{\infty}$, $\sigma_{zy}^{\infty}$ and remote electric displacements $D_x^{\infty}$ and $D_y^{\infty}$;

Case 3: remote mechanical strains $\gamma_{zx}^{\infty}$, $\gamma_{zy}^{\infty}$ and remote electric displacements $D_x^{\infty}$ and $D_y^{\infty}$;

Case 4: remote mechanical stresses $\sigma_{zx}^{\infty}$, $\sigma_{zy}^{\infty}$ and remote electric field strength $E_x^{\infty}$ and $E_y^{\infty}$.

Each case corresponds to a pair of $p_0$ and $q_0$, which are provided in Appendix 1. With the aid of the mapping function (9) and the following relation

$$\ln(1-\zeta) = -\sum_{k=1}^{\infty}\frac{\zeta^{k+1}}{k+1} \quad |\zeta|<1 \tag{20}$$

eqn (19) can be represented in a general form of Laurent series expansions around the inhomogeneity as

$$\left.\begin{aligned}\Psi_0(\zeta) &= \sum_{k=0}^{\infty}[a_k^0\zeta^{k+1}+b_k^0\zeta^{-(k+1)}]\\ \Phi_0(\zeta) &= \sum_{k=0}^{\infty}[c_k^0\zeta^{k+1}+d_k^0\zeta^{-(k+1)}]\end{aligned}\right\} \quad 1\leqslant|\zeta|<|\zeta_0| \tag{21}$$

where the constant terms denoting the equipotential field and the translation of a rigid body have been omitted. The coefficients $a_k^0$, $b_k^0$, $c_k^0$ and $d_k^0$ are given by the following:

$$a_k^0 = \begin{cases} -\dfrac{c_{44}^1 b_z}{2\pi i}\dfrac{1}{\zeta_0}+\dfrac{p_0 c}{2}R & k=0\\[2ex] -\dfrac{1}{k+1}\dfrac{c_{44}^1 b_z}{2\pi i}\dfrac{1}{\zeta_0^{k+1}} & k=1,2,\ldots,\end{cases} \tag{22a}$$

$$b_k^0 = \begin{cases} -\dfrac{c_{44}^1 b_z}{2\pi i}\dfrac{1}{R^2\zeta_0}+\dfrac{p_0 c}{2R} & k=0\\[2ex] -\dfrac{1}{k+1}\dfrac{c_{44}^1 b_z}{2\pi i}\left(\dfrac{1}{R^2\zeta_0}\right)^{k+1} & k=1,2,\ldots,\end{cases} \tag{22b}$$

$$c_k^0 = \begin{cases}\dfrac{q_0 c}{2}R & k=0\\ 0 & k=1,2,\ldots,\end{cases} \tag{22c}$$

$$d_k^0 = \begin{cases} \dfrac{q_0 c}{2R} & k = 0 \\ 0 & k = 1, 2, \ldots. \end{cases} \tag{22d}$$

Our task now is to determine the complex potentials $\Psi_j$ and $\Phi_j$ ($j = 1, 2$) for regions $\Omega_1$ and $\Omega_2$ which satisfy the continuity conditions (16) and (18).

## 3. GENERAL SOLUTIONS

Since in the $\zeta$-plane, $\Psi_1(\zeta)$ and $\Phi_1(\zeta)$ are holomorphic in the exterior of the unit circle $\Gamma_1$ and $\Psi_2(\zeta)$ and $\Phi_2(\zeta)$ are holomorphic in the annular region between the unit circle $\Gamma_1$ and the circle $\Gamma_2$ of radius $\rho = 1/R$ (Fig. 2), they can be expressed by Laurent's expansions, as follows:

$$\Psi_1(\zeta) = \sum_{k=0}^{\infty} b_k^1 \zeta^{-(k+1)}, \quad \Phi_1(\zeta) = \sum_{k=0}^{\infty} d_k^1 \zeta^{-(k+1)} \quad \zeta \in \Omega_1 \tag{23}$$

$$\left.\begin{aligned} \Psi_2(\zeta) &= \sum_{k=0}^{\infty} [a_k^2 \zeta^{k+1} + b_k^2 \zeta^{-(k+1)}] \\ \Phi_2(\zeta) &= \sum_{k=0}^{\infty} [c_k^2 \zeta^{k+1} + d_k^2 \zeta^{-(k+1)}] \end{aligned}\right\} \quad \zeta \in \Omega_2. \tag{24}$$

Substitution of eqn (24) into eqn (18) yields

$$a_k^2 = R^{2(k+1)} b_k^2, \quad c_k^2 = R^{2(k+1)} d_k^2. \tag{25}$$

Thus, eqn (24) can be rewritten as

$$\left.\begin{aligned} \Psi_2(\zeta) &= \sum_{k=0}^{\infty} [a_k^2 \zeta^{k+1} + a_k^2 R^{-2(k+1)} \zeta^{-(k+1)}] \\ \Phi_2(\zeta) &= \sum_{k=0}^{\infty} [c_k^2 \zeta^{k+1} + c_k^2 R^{-2(k+1)} \zeta^{-(k+1)}] \end{aligned}\right\} \quad \zeta \in \Omega_2. \tag{26}$$

Using Laurent series expansions (21) and (26) and noting that on the unit circle $\Gamma_1$ of Fig. 2, $\zeta = \sigma = 1/\bar{\sigma}$, the displacement continuity condition (16a) leads to

$$\sum_{k=0}^{\infty} \{\mu_1(a_k^0 + \bar{b}_k^0) - [a_k^2 + \bar{a}_k^2 R^{-2(k+1)}]\} \sigma^{k+1} + \mu_1 \overline{\Psi_1(1/\bar{\sigma})}$$
$$= \sum_{k=0}^{\infty} \{-\mu_1(\bar{a}_k^0 + b_k^0) + [\bar{a}_k^2 + a_k^2 R^{-2(k+1)}]\} \sigma^{-(k+1)} - \mu_1 \Psi_1(\sigma). \tag{27}$$

It is well known from the analytical function theory that if $\Psi(\zeta)$ is holomorphic in the exterior region $\Omega_1$ of a unit circle $\Gamma_1$ ($\rho = 1$), then $\overline{\Psi(1/\bar{\zeta})}$ is holomorphic in the interior region of the unit circle. Thus, if we let $\Omega_0$ represent the region inside the circle $\Gamma_2$ of radius $\rho = 1/R$ such that $\Omega_0 + \Omega_2$ occupies the whole region inside the unit circle, it follows from the above argument and eqn (27) that the function $\theta(\zeta)$ defined by

$$\theta(\zeta) = \begin{cases} \sum_{k=0}^{\infty} \{\mu_1(a_k^0+\bar{b}_k^0)-[a_k^2+\bar{a}_k^2 R^{-2(k+1)}]\}\zeta^{k+1}+\mu_1\overline{\Psi_1(1/\bar{\zeta})} & \zeta\in\Omega_2+\Omega_0 \\ \sum_{k=0}^{\infty} \{-\mu_1(\bar{a}_k^0+b_k^0)+[\bar{a}_k^2+a_k^2 R^{-2(k+1)}]\}\zeta^{-(k+1)}-\mu_1\Psi_1(\zeta) & \zeta\in\Omega_1 \end{cases} \tag{28}$$

will be holomorphic and single-valued in the entire plane. Hence, it is concluded, by Liouville's theorem, that $\theta(\zeta) \equiv$ constant. Since constant function $\Psi_1(\zeta)$ denotes rigid-body motion, which can be neglected, thus $\theta(\zeta) \equiv 0$. With this result, we have from eqns (28) and (23) that

$$\mu_1 b_k^1 = -\mu_1(\bar{a}_k^0+b_k^0)+\bar{a}_k^2+a_k^2 R^{-2(k+1)}. \tag{29}$$

Similarly, the remaining continuity conditions in eqn (16) produce

$$\mu_2 d_k^1 = -\mu_2(\bar{c}_k^0+d_k^0)+\bar{c}_k^2+c_k^2 R^{-2(k+1)} \tag{30}$$

$$b_k^1+\alpha_1 d_k^1 = \bar{a}_k^0-b_k^0+\alpha_1(\bar{c}_k^0-d_k^0)+(a_k^2+\alpha_2 c_k^2)R^{-2(k+1)}-(\bar{a}_k^2+\alpha_2\bar{c}_k^2) \tag{31}$$

$$\beta_1 b_k^1-d_k^1 = \beta_1(\bar{a}_k^0-b_k^0)-(\bar{c}_k^0-d_k^0)+(\beta_2 a_k^2-c_k^2)R^{-2(k+1)}-(\beta_2\bar{a}_k^2-\bar{c}_k^2). \tag{32}$$

The main task now is to determine the coefficients of the series expansion of the complex potentials. For a given $k$, eqns (29)–(32) provide a system of four linear equations with four unknowns $a_k^2$, $b_k^1$, $c_k^2$ and $d_k^1$. These unknown coefficients can be solved and expressed in terms of the specified coefficients $a_k^0$, $b_k^0$, $c_k^0$ and $d_k^0$ as

$$a_k^2 = I_k^{(1)}a_k^0+J_k^{(1)}\bar{a}_k^0+L_k^{(1)}c_k^0+N_k^{(1)}\bar{c}_k^0 \tag{33}$$

$$b_k^1 = I_k^{(2)}a_k^0+J_k^{(2)}\bar{a}_k^0+L_k^{(2)}c_k^0+N_k^{(2)}\bar{c}_k^0-b_k^0 \tag{34}$$

$$c_k^2 = I_k^{(3)}a_k^0+J_k^{(3)}\bar{a}_k^0+L_k^{(3)}c_k^0+N_k^{(3)}\bar{c}_k^0 \tag{35}$$

$$d_k^1 = I_k^{(4)}a_k^0+J_k^{(4)}\bar{a}_k^0+L_k^{(4)}c_k^0+N_k^{(4)}\bar{c}_k^0-d_k^0 \tag{36}$$

where the coefficients $I_k^{(n)}$, $J_k^{(n)}$, $L_k^{(n)}$ and $N_k^{(n)}$ ($n = 1,2,3,4$) have been given in Appendix 2.

Substituting eqn (22) into eqns (33)–(36), all the coefficients in the series expansions (23) and (24) for $\Psi_1(\zeta)$, $\Phi_1(\zeta)$, $\Psi_2(\zeta)$ and $\Phi_2(\zeta)$ are determined and the problem is thus solved. In some cases, the series solutions (23) and (24) can be given in simpler forms or summed up to obtain closed-form expressions. For example, in the absence of the dislocation, all the coefficients $a_k^2$, $b_k^1$, $c_k^2$ and $d_k^1$ vanish when $k \geqslant 1$. These solutions can then be given in closed forms in the physical $z$-plane as:

$$\Psi_0(z)+\Psi_1(z) = p_0 z+\tfrac{1}{2}[(I_0^{(2)}R^2-1)p_0+J_0^{(2)}R^2\bar{p}_0 + L_0^{(2)}R^2 q_0+N_0^{(2)}R^2\bar{q}_0][z-(z^2-c^2)^{1/2}] \quad z\in\Omega_1 \tag{37a}$$

$$\Phi_0(z)+\Phi_1(z) = q_0 z+\tfrac{1}{2}[I_0^{(4)}R^2 p_0+J_0^{(4)}R^2\bar{p}_0 + (L_0^{(4)}R^2-1)q_0+N_0^{(4)}R^2\bar{q}_0][z-(z^2-c^2)^{1/2}] \quad z\in\Omega_1 \tag{37b}$$

$$\Psi_2(z) = [I_0^{(1)}p_0+J_0^{(1)}\bar{p}_0+L_0^{(1)}q_0+N_0^{(1)}\bar{q}_0]z \quad z\in\Omega_2 \tag{38a}$$

$$\Phi_2(z) = [I_0^{(3)}p_0+J_0^{(3)}\bar{p}_0+L_0^{(3)}q_0+N_0^{(3)}\bar{q}_0]z \quad z\in\Omega_2. \tag{38b}$$

Substitution of the above solutions into eqn (7) produces the field components. It is easily found that since $\Psi_2(z)$ and $\Phi_2(z)$ are linear functions of $z$, the stress, the electric field strength and the electric displacement inside the elliptical inhomogeneity are uniform. The solutions (37) and (38) are in agreement with those derived by Zhong and Meguid (1997). In the next two sections, the interaction between a screw dislocation and a circular or an elliptical inhomogeneity will be examined and discussed.

## 4. INTERACTION BETWEEN SCREW DISLOCATION AND CIRCULAR INHOMOGENEITY

In the case of a circular inhomogeneity ($a = b$), the mapping function (9) becomes $z = \Omega(\zeta) = a\zeta$. Using relations (20) and (33)–(36), the field potentials $\Psi_1$, $\Phi_1$, $\Psi_2$ and $\Phi_2$ in eqns (23) and (24) can be obtained in closed forms and the solutions in $\Omega_1$ and $\Omega_2$ are given by

$$\Psi_0(z)+\Psi_1(z) = \frac{c_{44}^1 b_z}{2\pi i}\left[\ln(z-z_0)-\Delta_3\ln\left(1-\frac{a^2}{z\bar{z}_0}\right)\right]+p_0 z+(\bar{p}_0\Delta_3+\bar{q}_0\Delta_4)\frac{a^2}{z} \quad z\in\Omega_1 \tag{39a}$$

$$\Phi_0(z)+\Phi_1(z) = -\frac{c_{44}^1 b_z}{2\pi i}\Delta_7\ln\left(1-\frac{a^2}{z\bar{z}_0}\right)+q_0 z+(\bar{p}_0\Delta_7+\bar{q}_0\Delta_8)\frac{a^2}{z} \quad z\in\Omega_1 \tag{39b}$$

$$\Psi_2(z) = \frac{c_{44}^1 b_z}{2\pi i}\Delta_1\ln\left(1-\frac{z}{z_0}\right)+(p_0\Delta_1+q_0\Delta_2)z \quad z\in\Omega_2 \tag{40a}$$

$$\Phi_2(z) = \frac{c_{44}^1 b_z}{2\pi i}\Delta_5\ln\left(1-\frac{z}{z_0}\right)+(p_0\Delta_5+q_0\Delta_6)z \quad z\in\Omega_2 \tag{40b}$$

where

$$\Delta_1 = \frac{2c_{44}^2}{c_{44}^1}\frac{[c_{44}^1(\varepsilon_{11}^1+\varepsilon_{11}^2)+e_{15}^1(e_{15}^1+e_{15}^2)]}{\Delta}$$

$$\Delta_2 = \frac{2c_{44}^2}{\varepsilon_{11}^1}\frac{(e_{15}^1\varepsilon_{11}^2-\varepsilon_{11}^1 e_{15}^2)}{\Delta}$$

$$\Delta_3 = \frac{1}{\Delta}[(c_{44}^1-c_{44}^2)(\varepsilon_{11}^1+\varepsilon_{11}^2)+(e_{15}^1)^2-(e_{15}^2)^2]$$

$$\Delta_4 = \frac{2c_{44}^1}{\varepsilon_{11}^1}\frac{(e_{15}^1\varepsilon_{11}^2-\varepsilon_{11}^1 e_{15}^2)}{\Delta}$$

$$\Delta_5 = \frac{2\varepsilon_{11}^2}{c_{44}^1}\frac{(c_{44}^1 e_{15}^2-e_{15}^1 c_{44}^2)}{\Delta}$$

$$\Delta_6 = \frac{2\varepsilon_{11}^2}{\varepsilon_{11}^1}\frac{[\varepsilon_{11}^1(c_{44}^1+c_{44}^2)+e_{15}^1(e_{15}^1+e_{15}^2)]}{\Delta}$$

$$\Delta_7 = \frac{2\varepsilon_{11}^1}{c_{44}^1}\frac{(c_{44}^1 e_{15}^2-e_{15}^1 c_{44}^2)}{\Delta}$$

$$\Delta_8 = \frac{1}{\Delta}[(c_{44}^1+c_{44}^2)(\varepsilon_{11}^1-\varepsilon_{11}^2)+(e_{15}^1)^2-(e_{15}^2)^2]$$

with

$$\Delta = (c_{44}^1 + c_{44}^2)(\varepsilon_{11}^1 + \varepsilon_{11}^2) + (e_{15}^1 + e_{15}^2)^2.$$

The electric field strength, electric displacements and stresses, both in the matrix and the inhomogeneity, can be derived from eqns (7), (39) and (40) as follows:

$$E_{x1} - iE_{y1} = \frac{1}{\varepsilon_{11}^1}\left[\frac{c_{44}^1 b_z}{2\pi i}\Delta_7\left(\frac{1}{z - a^2/\bar{z}_0} - \frac{1}{z}\right) + (\bar{p}_0\Delta_7 + \bar{q}_0\Delta_8)\frac{a^2}{z^2} - q_0\right] \quad z \in \Omega_1$$

$$D_{x1} - iD_{y1} = \frac{c_{44}^1 b_z}{2\pi i}\left[\frac{e_{15}^1}{c_{44}^1}\frac{1}{z - z_0} - \left(\frac{e_{15}^1}{c_{44}^1}\Delta_3 - \Delta_7\right)\left(\frac{1}{z - a^2/\bar{z}_0} - \frac{1}{z}\right)\right] + \left(\frac{e_{15}^1}{c_{44}^1}p_0 - q_0\right)$$

$$- \left[\frac{e_{15}^1}{c_{44}^1}(\bar{p}_0\Delta_3 + \bar{q}_0\Delta_4) - (\bar{p}_0\Delta_7 + \bar{q}_0\Delta_8)\right]\frac{a^2}{z^2} \quad z \in \Omega_1$$

$$\sigma_{zx1} - i\sigma_{zy1} = \frac{c_{44}^1 b_z}{2\pi i}\left[\frac{1}{z - z_0} - \left(\Delta_3 + \frac{e_{15}^1}{\varepsilon_{11}^1}\Delta_7\right)\left(\frac{1}{z - a^2/\bar{z}_0} - \frac{1}{z}\right)\right] + \left(p_0 + \frac{e_{15}^1}{\varepsilon_{11}^1}q_0\right)$$

$$- \left[(\bar{p}_0\Delta_3 + \bar{q}_0\Delta_4) + \frac{e_{15}^1}{\varepsilon_{11}^1}(\bar{p}_0\Delta_7 + \bar{q}_0\Delta_8)\right]\frac{a^2}{z^2} \quad z \in \Omega_1$$

in the matrix, and

$$E_{x2} - iE_{y2} = -\frac{1}{\varepsilon_{11}^2}\left[\frac{c_{44}^1 b_z}{2\pi i}\Delta_5\frac{1}{z - z_0} + (p_0\Delta_5 + q_0\Delta_6)\right] \quad z \in \Omega_2$$

$$D_{x2} - iD_{y2} = \frac{c_{44}^1 b_z}{2\pi i}\left(\frac{e_{15}^2}{c_{44}^2}\Delta_1 - \Delta_5\right)\frac{1}{z - z_0} + \frac{e_{15}^2}{c_{44}^2}(p_0\Delta_1 + q_0\Delta_2) - (p_0\Delta_5 + q_0\Delta_6) \quad z \in \Omega_2$$

$$\sigma_{zx2} - i\sigma_{zy2} = \frac{c_{44}^1 b_z}{2\pi i}\left(\Delta_1 + \frac{e_{15}^2}{\varepsilon_{11}^2}\Delta_5\right)\frac{1}{z - z_0} + (p_0\Delta_1 + q_0\Delta_2) + \frac{e_{15}^2}{\varepsilon_{11}^2}(p_0\Delta_5 + q_0\Delta_6) \quad z \in \Omega_2$$

in the homogeneity.

The electro-elastic coupling effects induced by the dislocation can be evaluated by letting $p_0 = q_0 = 0$ in the above representations. It is easily seen that the electric field strength, both inside the inhomogeneity and the matrix, is influenced by the dislocation and will not vanish unless $\Delta_5 = \Delta_7 = 0$, i.e. that $e_{15}^1/c_{44}^1 = e_{15}^2/c_{44}^2$. The electric displacements, like the stresses, show classical screw dislocation behaviour with $1/(z - z_0)$ singularity at the point $z = z_0$. It will not vanish unless $e_{15}^1 = e_{15}^2 = 0$, which is the case when both the matrix and the inhomogeneity are elastic dielectric materials. It can also be found that in the absence of the dislocation, both the stress and the electric field strength are uniform in the inhomogeneity. This result was established by Pak (1992). In the absence of the electric fields, our solutions coincide with those of Smith (1968) and Gong and Meguid (1994).

If the inhomogeneity is replaced by a circular cavity, then $c_{44}^2 = e_{15}^2 = 0$ and $\varepsilon_{11}^2 = \varepsilon_0$. In this case, expressions (39) and (40) reduce to

$$\Psi_0(z)+\Psi_1(z) = \frac{c_{44}^1 b_z}{2\pi i}\left[\ln(z-z_0)-\ln\left(1-\frac{a^2}{z\bar{z}_0}\right)\right]+p_0 z$$

$$+\left\{\bar{p}_0+\bar{q}_0\frac{2c_{44}^1 e_{15}^1 \varepsilon_0}{\varepsilon_{11}^1[c_{44}^1(\varepsilon_{11}^1+\varepsilon_0)+(e_{15}^1)^2]}\right\}\frac{a^2}{z}$$

$$\Phi_0(z)+\Phi_1(z) = q_0 z+\bar{q}_0\frac{c_{44}^1(\varepsilon_{11}^1-\varepsilon_0)+(e_{15}^1)^2}{c_{44}^1(\varepsilon_{11}^1+\varepsilon_0)+(e_{15}^1)^2}\frac{a^2}{z}$$

$$\Psi_2(z) = 0$$

$$\Phi_2(z) = \frac{2\varepsilon_0[c_{44}^1\varepsilon_{11}^1+(e_{15}^1)^2]}{\varepsilon_{11}^1[c_{44}^1(\varepsilon_{11}^1+\varepsilon_0)+(e_{15}^1)^2]}q_0 z.$$

Apparently, the electric field strength, both in the cavity and the matrix, is not influenced by the dislocation and the remote equivalent mechanical field $p_0$. In addition, the electric field strength and the electric displacement are uniform inside the cavity.

## 5. INTERACTION BETWEEN SCREW DISLOCATION AND ELLIPTICAL INHOMOGENEITY

### 5.1. *Field solutions for elastic problems*

For the elliptical inhomogeneity, no closed-form field potentials exist. In the absence of the electric fields, however, the problem becomes a purely elastic one, and the expressions (33)–(36) for the coefficients, $a_k^2$, $b_k^1$, $c_k^2$ and $d_k^1$ can be given by the following simple forms

$$a_k^2 = I_k^{(1)}a_k^0+J_k^{(1)}\bar{a}_k^0, \quad c_k^2 = 0,$$
$$b_k^1 = I_k^{(2)}a_k^0+J_k^{(2)}\bar{a}_k^0-b_k^0, \quad d_k^1 = 0. \tag{41}$$

It follows from eqns (21)–(23), (26) and (41) that

$$\Psi_0(\zeta)+\Psi_1(\zeta) = \frac{c_{44}^1 b_z}{2\pi i}\ln\left(1-\frac{\zeta}{\zeta_0}\right)+\frac{c}{2}p_0 R\zeta+\frac{c}{2}(I_0^{(2)}p_0+J_0^{(2)}\bar{p}_0)\frac{R}{\zeta}$$
$$-\frac{c_{44}^1 b_z}{2\pi i}\sum_{k=0}^{\infty}\left(\frac{1}{k+1}\right)[I_k^{(2)}\zeta_0^{-(k+1)}-J_k^{(2)}\bar{\zeta}_0^{-(k+1)}]\zeta^{-(k+1)} \quad \zeta\in\Omega_1 \tag{42}$$

$$\Psi_2(\zeta) = -\frac{c_{44}^1 b_z}{2\pi i}\sum_{k=0}^{\infty}\left\{\left(\frac{1}{k+1}\right)[I_k^{(1)}(R\zeta_0)^{-(k+1)}-J_k^{(1)}(R\bar{\zeta}_0)^{-(k+1)}]\right.$$
$$\left.\times[(R\zeta)^{(k+1)}+(R\zeta)^{-(k+1)}]\right\}+(I_0^{(1)}p_0+J_0^{(1)}\bar{p}_0)z \quad \zeta\in\Omega_2 \tag{43}$$

where

$$I_k^{(1)} = \frac{2\mu_1(1+\mu_1)}{(1+\mu_1)^2-(1-\mu_1)^2R^{-4(k+1)}}, \quad J_k^{(1)} = \frac{-2\mu_1(1-\mu_1)R^{-2(k+1)}}{(1+\mu_1)^2-(1-\mu_1)^2R^{-4(k+1)}}$$
$$I_k^{(2)} = \frac{4\mu_1 R^{-2(k+1)}}{(1+\mu_1)^2-(1-\mu_1)^2R^{-4(k+1)}}, \quad J_k^{(2)} = \frac{(1-\mu_1^2)[1-R^{-4(k+1)}]}{(1+\mu_1)^2-(1-\mu_1)^2R^{-4(k+1)}}.$$

If the dislocation is located at a point $z_0 = \Omega_0(\zeta_0)$ along the $x$-axis and the remote strain

$\gamma_{zy}^{\infty}$ or stress $\sigma_{zy}^{\infty}$ vanishes, such that $\zeta_0 = \bar{\zeta}_0$ and $p_0 = \bar{p}_0$, the field solutions (42) and (43) become

$$\Psi_0(\zeta)+\Psi_1(\zeta) = \frac{c_{44}^1 b_z}{2\pi i}\ln\left(1-\frac{\zeta}{\zeta_0}\right)+\frac{c}{2}p_0 R\zeta + p_0\frac{(1+\mu_1)+(1-\mu_1)R^2}{(1-\mu_1)+(1+\mu_1)R^2}\frac{cR}{2\zeta}$$
$$+\frac{c_{44}^1 b_z}{2\pi i}\sum_{k=0}^{\infty}\left(\frac{1}{k+1}\right)\frac{(1+\mu_1)-(1-\mu_1)R^{2(k+1)}}{(1-\mu_1)-(1+\mu_1)R^{2(k+1)}}(\zeta\zeta_0)^{-(k+1)} \quad \zeta\in\Omega_1 \quad (44)$$

$$\Psi_2(\zeta) = \frac{c_{44}^1 b_z}{2\pi i}\sum_{k=0}^{\infty}\left\{\left(\frac{1}{k+1}\right)\frac{2\mu_1 R^{k+1}\zeta_0^{-(k+1)}}{(1-\mu_1)-(1+\mu_1)R^{2(k+1)}}[(R\zeta)^{(k+1)}+(R\zeta)^{-(k+1)}]\right\}$$
$$+\frac{2\mu_1 R^2 p_0}{(1-\mu_1)+(1+\mu_1)R^2}z \quad \zeta\in\Omega_2. \quad (45)$$

The same problem has been considered, in the absence of $p_0$, by Sendeckyj (1970). In terms of our notation, his result can be expressed as

$$\Psi_0(\zeta)+\Psi_1(\zeta) = \frac{c_{44}^1 b_z}{2\pi i}\left\{\ln(\zeta-\zeta_0)+K\ln\left(1-\frac{1}{\zeta\zeta_0}\right)\right.$$
$$\left.+(1-K^2)\sum_{n=0}^{\infty}\left[(-K)^n\ln\left(1-\frac{1}{\zeta\zeta_0 R^{2n+2}}\right)\right]\right\} \quad (46)$$

$$\Psi_2(\zeta) = \frac{c_{44}^1 b_z}{2\pi i}(1+K)\sum_{n=0}^{\infty}(-K)^n\ln[z-\Omega(R^{2n}\zeta_0)] \quad (47)$$

where $K=(\mu_1-1)/(\mu_1+1)$ and $\Omega(\zeta)$ is the mapping function defined in eqn (9). In the absence of $p_0$, our results (44) and (45) are identical to eqns (46) and (47), except for a constant term representing the rigid-body displacement. The equivalence between eqns (44) and (46) or (45) and (47) can be easily established by expanding eqns (46) or (47) into a double series form and noting that one of the series can be summed up to give closed-form expressions.

### 5.2. *Interaction between screw dislocation and elliptical cavity*

If the inhomogeneity is an elliptical cavity, we have $c_{44}^2 = 0$, $e_{15}^2 = 0$, and $\varepsilon_{11}^2 = \varepsilon_0$. In this case, the field solutions become

$$\Psi_0(\zeta)+\Psi_1(\zeta) = \frac{c_{44}^1 b_z}{2\pi i}\left[\ln\left(1-\frac{\zeta}{\zeta_0}\right)-\ln\left(1-\frac{1}{\bar{\zeta}_0\zeta}\right)\right]+p_0 z$$
$$+\frac{1}{2}(-p_0+R^2\bar{p}_0+L_0^{(2)}R^2 q_0+N_0^{(2)}R^2\bar{q}_0)[z-(z^2-c^2)^{1/2}] \quad (48a)$$

$$\Phi_0(\zeta)+\Phi_1(\zeta) = q_0 z+\tfrac{1}{2}[(L_0^{(4)}R^2-1)q_0+N_0^{(4)}R^2\bar{q}_0][z-(z^2-c^2)^{1/2}] \quad (48b)$$

$$\Psi_2(\zeta) = 0 \quad (49a)$$

$$\Phi_2(\zeta) = [L_0^{(3)}q_0+N_0^{(3)}\bar{q}_0]z \quad (49b)$$

where

$$L_0^{(2)} = -4\mu_2\alpha_1(1+\alpha_1\beta_1)R^2/\delta$$

$$N_0^{(2)} = 2\mu_2\alpha_1[(1+\alpha_1\beta_1-\mu_2)+(1+\alpha_1\beta_1+\mu_2)R^4]/\delta$$

$$L_0^{(3)} = 2\mu_2(1+\alpha_1\beta_1)(1+\alpha_1\beta_1+\mu_2)R^4/\delta$$

$$N_0^{(3)} = -2\mu_2(1+\alpha_1\beta_1)(1+\alpha_1\beta_1-\mu_2)R^2/\delta$$

$$L_0^{(4)} = 4\mu_2(1+\alpha_1\beta_1)R^2/\delta$$

$$N_0^{(4)} = \{2(1+\alpha_1\beta_1)[(1+\alpha_1\beta_1+\mu_2)R^4-(1+\alpha_1\beta_1-\mu_2)]/\delta\}-1$$

with

$$\delta = (1+\alpha_1\beta_1+\mu_2)^2R^4-(1+\alpha_1\beta_1-\mu_2)^2.$$

It follows from eqns (7), (48) and (49) that

$$E_{x1}-iE_{y1} = -\frac{q_0}{\varepsilon_{11}^1}+\frac{1}{\varepsilon_{11}^1}[(L_0^{(4)}R^2-1)q_0+N_0^{(4)}R^2\bar{q}_0]\frac{1}{R^2\zeta^2-1} \quad \zeta\in\Omega_1$$

$$\begin{aligned} D_{x1}-iD_{y1} = {} & \frac{e_{15}^1 b_z}{2\pi i}\left(\frac{1}{\zeta-\zeta_0}+\frac{1}{\zeta}-\frac{1}{\zeta-1/\bar{\zeta}_0}\right)\frac{2R\zeta^2}{c(R^2\zeta^2-1)}+\left(\frac{e_{15}^1}{c_{44}^1}p_0-q_0\right) \\ & -\left\{\frac{e_{15}^1}{c_{44}^1}[-p_0+R^2\bar{p}_0+L_0^{(2)}R^2q_0+N_0^{(2)}R^2\bar{q}_0]\right. \\ & \left.-[(L_0^{(4)}R^2-1)q_0+N_0^{(4)}R^2\bar{q}_0]\right\}\frac{1}{R^2\zeta^2-1} \quad \zeta\in\Omega_1 \end{aligned}$$

$$\begin{aligned} \sigma_{zx1}-i\sigma_{zy1} = {} & \frac{c_{44}^1 b_z}{2\pi i}\left(\frac{1}{\zeta-\zeta_0}+\frac{1}{\zeta}-\frac{1}{\zeta-1/\bar{\zeta}_0}\right)\frac{2R\zeta^2}{c(R^2\zeta^2-1)}+\left(p_0+\frac{e_{15}^1}{\varepsilon_{11}^1}q_0\right) \\ & -\left\{[-p_0+R^2\bar{p}_0+L_0^{(2)}R^2q_0+N_0^{(2)}R^2\bar{q}_0]\right. \\ & \left.+\frac{e_{15}^1}{\varepsilon_{11}^1}[(L_0^{(4)}R^2-1)q_0+N_0^{(4)}R^2\bar{q}_0]\right\}\frac{1}{R^2\zeta^2-1} \quad \zeta\in\Omega_1 \end{aligned}$$

in the matrix, and

$$E_{x2}-iE_{y2} = -\frac{1}{\varepsilon_0}(L_0^{(3)}q_0+N_0^{(3)}\bar{q}_0) \quad \zeta\in\Omega_2$$

$$D_{x2}-iD_{y2} = -(L_0^{(3)}q_0+N_0^{(3)}\bar{q}) \quad \zeta\in\Omega_2$$

in the cavity.

It is clear that the electric displacements exhibit $1/r$ ($r = \zeta-\zeta_0$) singularity at the point $\zeta = \zeta_0$ in the matrix and are uniform in the cavity. In addition, the electric field strength, both inside and outside the cavity, are not affected by the dislocation and the remote equivalent mechanical field $p_0$, and are uniform inside the cavity.

If we let the ratio of the major and minor diameters of the elliptical cavity approaches zero such that the cavity can be taken as a slit crack, it follows from eqn (10) that $\varepsilon = b/a \to 0$, $R \to 1$, and $c \to a$. In this case, expressions (48) and (49) reduce to

$$\Psi_0(\zeta)+\Psi_1(\zeta) = \frac{c^1_{44}b_z}{2\pi i}\left[\ln\left(1-\frac{\zeta}{\zeta_0}\right)-\ln\left(1-\frac{1}{\bar{\zeta}_0\zeta}\right)\right]+p_0 z$$

$$-i\,\mathrm{Im}\left(p_0+\frac{e^1_{15}}{\varepsilon^1_{11}}q_0\right)[z-(z^2-a^2)^{1/2}] \quad (50a)$$

$$\Phi_0(\zeta)+\Phi_1(\zeta) = q_0 z \quad (50b)$$

$$\Phi_2(\zeta) = \left\{\frac{\varepsilon_0}{\varepsilon^1_{11}}\mathrm{Re}\,q_0+i\frac{\varepsilon^1_{11}c^1_{44}+(e^1_{15})^2}{\varepsilon^1_{11}c^1_{44}}\mathrm{Im}\,q_0\right\}z. \quad (51)$$

It is interesting to note from the above equations that the electric field strength, either in the matrix or along the crack faces, is uniform and can be expressed as

$$E_x-iE_y = -\frac{1}{\varepsilon^1_{11}}q_0 \quad \text{in the matrix}$$

$$E_x-iE_y = -\frac{1}{\varepsilon_0}\left[\frac{\varepsilon_0}{\varepsilon^1_{11}}\mathrm{Re}\,q_0+i\frac{\varepsilon^1_{11}c^1_{44}+(e^1_{15})^2}{\varepsilon^1_{11}c^1_{44}}\mathrm{Im}\,q_0\right] \quad \text{along the crack faces.}$$

The stresses and electric displacements, however, show traditional square root singularities near the ends of the slit crack. This phenomenon was also observed by Pak and Tobin (1993) in the absence of the dislocation. The above results are very important in determining the electric boundary conditions of a cavity or a crack problem in piezoelectric media.

## 6. CONCLUSIONS

The electro-elastic interaction between a screw dislocation and an elliptical piezoelectric inhomogeneity in an infinite piezoelectric material is investigated. By using conformal mapping and the perturbation technique, the general series solutions for the field potentials in both the inhomogeneity and the surrounding matrix are obtained *explicitly*. In the case of a circular inclusion or an elliptical cavity, closed-form field potentials are derived. It is found that when the inhomogeneity reduces to a cavity, the electric field strength, both inside and outside the cavity, attains a uniform distribution in the cavity and is not affected by the presence of a dislocation in the matrix. If the elliptical cavity further reduces to a slit crack, both the stress and the electric displacement exhibit traditional square root singularities, while the electric field strength becomes uniform in the matrix and along the crack faces.

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## APPENDIX 1

*Expressions for complex constants* $p_0$ *and* $q_0$ *corresponding to different combinations of remote electric and mechanical loads*

The complex constants $p_0$ and $q_0$ in eqn (19) can be determined from the following four cases of the boundary conditions given at infinity:

(Case 1) remote mechanical strains $\gamma_{zx}^{\infty}$, $\gamma_{zy}^{\infty}$ and remote electric field strength $E_x^{\infty}$ and $E_y^{\infty}$ will yield

$$p_0 = c_{44}^1\gamma_{zx}^{\infty} - ic_{44}^1\gamma_{zy}^{\infty}, \quad q_0 = -\varepsilon_{11}^1 E_x^{\infty} + i\varepsilon_{11}^1 E_y^{\infty}; \tag{A1}$$

(Case 2) remote mechanical stresses $\sigma_{zx}^{\infty}$, $\sigma_{zy}^{\infty}$ and remote electric displacements $D_x^{\infty}$ and $D_y^{\infty}$ will yield

$$p_0 = \frac{\sigma_{zx}^{\infty} + (e_{15}^1/c_{44}^1)D_x^{\infty}}{1+(e_{15}^1)^2/(\varepsilon_{11}^1 c_{44}^1)} - i\frac{\sigma_{zy}^{\infty} + (e_{15}^1/c_{44}^1)D_y^{\infty}}{1+(e_{15}^1)^2/(\varepsilon_{11}^1 c_{44}^1)},$$

$$q_0 = \frac{(e_{15}^1/c_{44}^1)\sigma_{zx}^{\infty} - D_x^{\infty}}{1+(e_{15}^1)^2/(\varepsilon_{11}^1 c_{44}^1)} - i\frac{(e_{15}^1/c_{44}^1)\sigma_{zy}^{\infty} - D_y^{\infty}}{1+(e_{15}^1)^2/(\varepsilon_{11}^1 c_{44}^1)}; \tag{A2}$$

(Case 3) remote mechanical strain $\gamma_{zx}^{\infty}$, $\gamma_{zy}^{\infty}$ and remote electric displacements $D_x^{\infty}$ and $D_y^{\infty}$ will yield

$$p_0 = c_{44}^1\gamma_{zx}^{\infty} - ic_{44}^1\gamma_{zy}^{\infty}, \quad q_0 = (e_{15}^1\gamma_{zx}^{\infty} - D_x^{\infty}) - i(e_{15}^1\gamma_{zy}^{\infty} - D_y^{\infty}); \tag{A3}$$

(Case 4) remote mechanical stresses $\sigma_{zx}^{\infty}$, $\sigma_{zy}^{\infty}$ and remote electric field strength $E_x^{\infty}$ and $E_y^{\infty}$ will yield

$$p_0 = (\sigma_{zx}^{\infty} + e_{15}^1 E_x^{\infty}) - i(\sigma_{zy}^{\infty} + e_{15}^1 E_y^{\infty}), \quad q_0 = -\varepsilon_{11}^1 E_x^{\infty} + i\varepsilon_{11}^1 E_y^{\infty}. \tag{A4}$$

## APPENDIX 2

*Details of coefficients in eqns* (33)–(36)

The coefficients in eqns (33)–(36) are given as follows:

$$I_k^{(1)} = R^{2(k+1)}\left(\frac{\lambda_{1,k}}{\delta_{1,k}} + \frac{\lambda_{3,k}}{\delta_{2,k}}\right), \quad J_k^{(1)} = R^{2(k+1)}\left(\frac{\lambda_{1,k}}{\delta_{1,k}} - \frac{\lambda_{3,k}}{\delta_{2,k}}\right),$$

$$L_k^{(1)} = R^{2(k-1)}\left(\frac{\lambda_{2,k}}{\delta_{1,k}} + \frac{\lambda_{4,k}}{\delta_{2,k}}\right), \quad N_k^{(1)} = R^{2(k+1)}\left(\frac{\lambda_{2,k}}{\delta_{1,k}} - \frac{\lambda_{4,k}}{\delta_{2,k}}\right). \tag{A5}$$

$$I_k^{(3)} = R^{2(k+1)}\left(\frac{\lambda_{5,k}}{\delta_{1,k}}+\frac{\lambda_{7,k}}{\delta_{2,k}}\right),\quad J_k^{(3)} = R^{2(k+1)}\left(\frac{\lambda_{5,k}}{\delta_{1,k}}-\frac{\lambda_{7,k}}{\delta_{2,k}}\right),$$

$$L_k^{(3)} = R^{2(k+1)}\left(\frac{\lambda_{6,k}}{\delta_{1,k}}+\frac{\lambda_{8,k}}{\delta_{2,k}}\right),\quad N_k^{(3)} = R^{2(k+1)}\left(\frac{\lambda_{6,k}}{\delta_{1,k}}-\frac{\lambda_{8,k}}{\delta_{2,k}}\right). \tag{A6}$$

$$I_k^{(2)} = \frac{1+R^{2(k+1)}}{\mu_1}\frac{\lambda_{1,k}}{\delta_{1,k}}+\frac{1-R^{2(k+1)}}{\mu_1}\frac{\lambda_{3,k}}{\delta_{2,k}},\quad J_k^{(2)} = \frac{1+R^{2(k+1)}}{\mu_1}\frac{\lambda_{1,k}}{\delta_{1,k}}-\frac{1-R^{2(k+1)}}{\mu_1}\frac{\lambda_{3,k}}{\delta_{2,k}}-1,$$

$$L_k^{(2)} = \frac{1+R^{2(k+1)}}{\mu_1}\frac{\lambda_{2,k}}{\delta_{1,k}}+\frac{1-R^{2(k+1)}}{\mu_1}\frac{\lambda_{4,k}}{\delta_{2,k}},\quad N_k^{(2)} = \frac{1+R^{2(k+1)}}{\mu_1}\frac{\lambda_{2,k}}{\delta_{1,k}}-\frac{1-R^{2(k+1)}}{\mu_1}\frac{\lambda_{4,k}}{\delta_{2,k}}. \tag{A7}$$

$$I_k^{(4)} = \frac{1+R^{2(k+1)}}{\mu_2}\frac{\lambda_{5,k}}{\delta_{1,k}}+\frac{1-R^{2(k+1)}}{\mu_2}\frac{\lambda_{7,k}}{\delta_{2,k}},\quad J_k^{(4)} = \frac{1+R^{2(k-1)}}{\mu_2}\frac{\lambda_{5,k}}{\delta_{1,k}}-\frac{1-R^{2(k-1)}}{\mu_2}\frac{\lambda_{7,k}}{\delta_{2,k}},$$

$$L_k^{(4)} = \frac{1+R^{2(k+1)}}{\mu_2}\frac{\lambda_{6,k}}{\delta_{1,k}}+\frac{1-R^{2(k+1)}}{\mu_2}\frac{\lambda_{8,k}}{\delta_{2,k}},\quad N_k^{(4)} = \frac{1+R^{2(k+1)}}{\mu_2}\frac{\lambda_{6,k}}{\delta_{1,k}}-\frac{1-R^{2(k+1)}}{\mu_2}\frac{\lambda_{8,k}}{\delta_{2,k}}-1. \tag{A8}$$

Where

$$\lambda_{1,k} = -\left[\left(\frac{1}{\mu_2}+1\right)R^{2(k+1)}+\left(\frac{1}{\mu_2}-1\right)\right]-\beta_1\left[\left(\frac{\alpha_1}{\mu_2}+\alpha_2\right)R^{2(k+1)}+\left(\frac{\alpha_1}{\mu_2}-\alpha_2\right)\right] \tag{A9}$$

$$\lambda_{2,k} = (\alpha_1-\alpha_2)(1-R^{2(k+1)}) \tag{A10}$$

$$\lambda_{3,k} = -\left[\left(\frac{1}{\mu_2}+1\right)R^{2(k-1)}-\left(\frac{1}{\mu_2}-1\right)\right]-\beta_1\left[\left(\frac{\alpha_1}{\mu_2}+\alpha_2\right)R^{2(k+1)}-\left(\frac{\alpha_1}{\mu_2}-\alpha_2\right)\right] \tag{A11}$$

$$\lambda_{4,k} = (\alpha_2-\alpha_1)(1+R^{2(k+1)}) \tag{A12}$$

$$\lambda_{5,k} = (\beta_2-\beta_1)(1-R^{2(k+1)}) \tag{A13}$$

$$\lambda_{6,k} = -\left[\left(\frac{1}{\mu_1}+1\right)R^{2(k+1)}+\left(\frac{1}{\mu_1}-1\right)\right]-\alpha_1\left[\left(\frac{\beta_1}{\mu_1}+\beta_2\right)R^{2(k+1)}+\left(\frac{\beta_1}{\mu_1}-\beta_2\right)\right] \tag{A14}$$

$$\lambda_{7,k} = (\beta_1-\beta_2)(1+R^{2(k+1)}) \tag{A15}$$

$$\lambda_{8,k} = -\left[\left(\frac{1}{\mu_1}+1\right)R^{2(k+1)}-\left(\frac{1}{\mu_1}-1\right)\right]-\alpha_1\left[\left(\frac{\beta_1}{\mu_1}+\beta_2\right)R^{2(k+1)}-\left(\frac{\beta_1}{\mu_1}-\beta_2\right)\right] \tag{A16}$$

with

$$\delta_{1,k} = -\left[\left(\frac{1}{\mu_1}+1\right)R^{2(k+1)}+\left(\frac{1}{\mu_1}-1\right)\right]\left[\left(\frac{1}{\mu_2}+1\right)R^{2(k+1)}+\left(\frac{1}{\mu_2}-1\right)\right]$$
$$-\left[\left(\frac{\beta_1}{\mu_1}+\beta_2\right)R^{2(k+1)}+\left(\frac{\beta_1}{\mu_1}-\beta_2\right)\right]\left[\left(\frac{\alpha_1}{\mu_2}+\alpha_2\right)R^{2(k-1)}+\left(\frac{\alpha_1}{\mu_2}-\alpha_2\right)\right] \tag{A17}$$

$$\delta_{2,k} = -\left[\left(\frac{1}{\mu_1}+1\right)R^{2(k+1)}-\left(\frac{1}{\mu_1}-1\right)\right]\left[\left(\frac{1}{\mu_2}+1\right)R^{2(k+1)}-\left(\frac{1}{\mu_2}-1\right)\right]$$
$$-\left[\left(\frac{\beta_1}{\mu_1}+\beta_2\right)R^{2(k+1)}-\left(\frac{\beta_1}{\mu_1}-\beta_2\right)\right]\left[\left(\frac{\alpha_1}{\mu_2}+\alpha_2\right)R^{2(k+1)}-\left(\frac{\alpha_1}{\mu_2}-\alpha_2\right)\right] \tag{A18}$$

and $\mu_1$, $\mu_2$, $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ given in eqn (17).